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Journal of Sound and Vibration 281 (2005) 1023–1035
JOURNAL OF SOUND AND VIBRATION
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A transient response analysis in the state-space applying the average velocity concept

Byung Ok Kim, An Sung Lee*

Rotor Dynamics Group, Korea Institute of Machinery and Materials, Yoosung, P.O. Box 101, Daejeon, 305-600, Republic of Korea

Received 10 November 2003; received in revised form 5 February 2004; accepted 12 February 2004
Available online 25 September 2004

Abstract

An implicit direct-time integration method for obtaining transient responses of general dynamic systems is described. The conventional Newmark method cannot be directly applied to state-space first-order differential equations, which contain no explicit acceleration terms. The method proposed here is the state-space Newmark method that incorporates the average velocity concept, and can be applied to an analysis of general dynamic systems that are expressed by state-space first-order differential equations. It is also readily coded into a program. Stability and accuracy analyses indicate that the method is numerically unconditionally stable like the conventional Newmark method, and has a period error of second-order accuracy for small damping and fourth-order for large damping and an amplitude error of second-order, regardless of damping. In addition, its utility and validity are confirmed by two application examples. The results suggest that the proposed state-space Newmark method based on average velocity be generally applied to the analysis of transient responses of general dynamic systems with a high degree of reliability with respect to stability and accuracy.

1. Introduction

Generally, available direct-time integration methods for transient response analyses are classified as the explicit time integration procedure, e.g., the central difference and Runge–Kutta

*Corresponding author. Tel: +82-42-868-7356, Fax: +082-42-868-7440.
E-mail addresses: kbo2612@kimm.re.kr (B.O. Kim), aslee@kimm.re.kr (A.S. Lee).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2004.02.003

methods, and the implicit time integration procedure, e.g., Newmark, Wilson-θ and Houbolt methods [1–6].

Most previous studies of the direct-time integration methods had focused on improvements in computational efficiency and accuracy by improving existing conventional methods or adapting them for specific applications. Trujillo [7], Park and Houser [8], Braekhus and Aasen [9] and Bhatti and Pister [10] presented advanced explicit time integration schemes intended for applications to general structures and structures with material nonlinearities. Klein and Trujillo [11], Zienkiewicz [12], Hilber and Hughes [13] and Wood et al. [14] described improved implicit methods for transient response analyses. Belyschko et al. [15], Belyschko and Mullen [16], Hughes and Liu [17] and Bazzi and Anderheggen [18] reported on direct-time integration procedures that combine explicit and implicit methods, and investigated their numerical stabilities and performed transient analyses. Hughes [19], Dokainish and Subbaraj [20] and Subbaraj and Dokainish [21] presented comprehensive surveys of direct-time integration methods, such as conventional explicit and implicit methods, improved implicit methods and the mixed implicit–explicit method, and summarized the computational procedures involved in transient analyses. Subbiah et al. [22] and Ratan and Rodriguez [23,24] also proposed time integration algorithms that made use of the explicit and implicit methods to predict the transient behaviors of conservative, non conservative and nonlinear rotor-bearing systems. Recently, some time integration methods that are unconditionally stable and have higher-order accuracies have been reported. Among these, Kim et al. [25] proposed a method obtained by the generalized average acceleration concept and Fung [26,27] described a method obtained by applying the collocation technique to first- and second-order differential equations.

Generally, explicit methods have some advantages over implicit methods in terms of computational cost and memory space per calculation step but they should have a very fine time-step size, Δt to ensure their numerical stability [4]. In reality, for explicit methods Δt acts as quite a stringent condition for stability, whereas implicit methods are generally stable for any given Δt and even their accuracy is guaranteed for an appropriate Δt. Among implicit methods, a family of the Newmark methods are the most popular and based on the average acceleration the Newmark method, which is unconditionally stable and has a second-order accuracy is the most widely used [5].

For a general dynamic system its damping is not proportional to its inertia or stiffness, and as a result, it is not possible to directly solve the eigenvalue problem. In this case, an eigenvalue analysis can be performed by transforming the system equation into a first-order differential equation by introducing the state-space vector [28]. In particular, for a rotor-bearing system, due to its gyroscopic effect and asymmetrical bearing stiffness and damping, a state-space first-order differential equation is frequently used for the eigenvalue analysis. However, since the state-space equation contains no explicit acceleration term, the above-mentioned conventional average acceleration Newmark method cannot be applied directly to a transient response analysis.

As an implicit direct-time integration method, this study presents the state-space Newmark method in which the average velocity concept is introduced, which can be directly applied to transient response analyses of general dynamic systems expressed by state-space first-order differential equations. Its formulation is presented, and the stability and accuracy of numerical solutions are analyzed and evaluated. The utility and validity of the method are also illustrated by applications to the transient analyses of two sample systems.

2. State-space Newmark method

2.1. General formulation

By introducing the state-space vector, a second-order differential equation of general dynamic system can be rewritten by a state-space first-order differential equation as given by

$$\{\dot{q}\} = [A]\{q\} + \{F\}, \tag{1}$$

where $\{q\} = [\{\dot{x}\}, \{x\}]^{\mathrm{T}}$ is the state-space vector, $\{x\}$ the displacement vector, $[A]$ the system dynamic matrix and $\{F\}$ the forcing vector, and $[A]$ and $\{F\}$ are given by

$$[A] = \begin{bmatrix} -[M]^{-1}[C] & -[M]^{-1}[K] \\ [I] & [0] \end{bmatrix}, \tag{2}$$

$$\{F\} = \begin{Bmatrix} [M]^{-1}\{f(t)\} \\ \{0\} \end{Bmatrix}, \tag{3}$$

where $[M]$, $[C]$, $[K]$ are the inertia, damping and stiffness matrices of the system, and $f(t)$ the external forcing vector. Referring to Fig. 1, defining a new variable $\tau\,(0 \leqslant \tau \leqslant \Delta t)$ for the time interval Δt between time t_{n+1} and t_n (or between the step n+1 and n) and introducing the constant average velocity, $\{\dot{q}(\tau)\}$, within Δt, then

$$\{\dot{q}(\tau)\} = \frac{1}{2}\left[\{\dot{q}\}_{n+1} + \{\dot{q}\}_n\right], \tag{4}$$

Considering $\{q(\tau)\} = \{q\}_n$ at $\tau = 0$ and integrating Eq. (4),

$$\{q(\tau)\} = \{q\}_n + \frac{\tau}{2}\left[\{\dot{q}\}_{n+1} + \{\dot{q}\}_n\right], \tag{5}$$

From Eq. (5) the displacement vector at $\tau = \Delta t$ or $t = t_{n+1}$ is

$$\{q\}_{n+1} = \{q\}_n + \frac{\Delta t}{2}\left[\{\dot{q}\}_{n+1} + \{\dot{q}\}_n\right]. \tag{6}$$

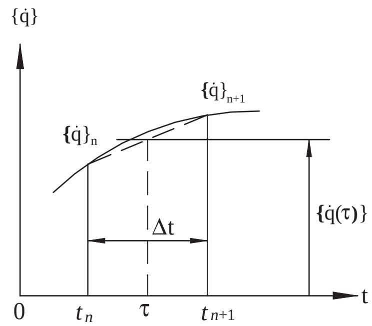

Fig. 1. Constant average velocity for the state-space Newmark method.

From Eq. (6) the velocity vector at time t_{n+1} is

$$\{\dot{q}\}_{n+1} = \frac{2}{\Delta t}\left[\{q\}_{n+1} - \{q\}_n\right] - \{\dot{q}\}_n. \tag{7}$$

Finally, considering Eq. (1) at time t_{n+1} and substituting it into Eq. (7),

$$\{q\}_{n+1} = \left([I] - \frac{\Delta t}{2}[A]\right)^{-1}\left(\{q\}_n + \frac{\Delta t}{2}\{\dot{q}\}_n + \frac{\Delta t}{2}\{F\}_{n+1}\right). \tag{8}$$

Now, the state-space vector values at time t_{n+1} can be readily calculated from Eq. (8) using the values at time t_n. The formulation of the above-proposed average velocity state-space Newmark method is much simpler and more straightforward than that of the conventional average acceleration Newmark method. Therefore, it can be more readily coded into a software program.

2.2. Stability and accuracy analyses

Here, stability and accuracy analyses of the proposed average velocity state-space Newmark method are performed, considering a 1-dof system for the sake of convenience. However, results are valid for general multidegrees of freedom systems without any loss in generality [26].

2.2.1. Stability analysis of solution

Considering Eq. (1) at time t_n and excluding the forcing vector, then

$$\{\dot{q}\}_n = [A]\{q\}_n. \tag{9}$$

Substituting Eq. (9) into Eq. (8) and constructing the recursive relationship, the state-space vector at time $t_n + p\Delta t$ is related to that at time t_n by

$$\{q\}_{n+p} = [A_m]^p\{q\}_n, \quad p = 0, 1, 2, \ldots, \tag{10}$$

where $[A_m]$ is the amplification matrix given by Eq. (11) with $\omega = \sqrt{k/m}$ and $\zeta = c/2\sqrt{km}$, and m, c, k are the mass, damping, stiffness of a 1-dof system.

$$\begin{aligned}[A_m] &= \left([I] - \frac{\Delta t}{2}[A]\right)^{-1}\left([I] + \frac{\Delta t}{2}[A]\right) \\ &= \begin{bmatrix} \frac{4-4\zeta\omega\Delta t-\omega^2\Delta t^2}{4+4\zeta\omega\Delta t+\omega^2\Delta t^2} & \frac{-4\omega^2\Delta t}{4+4\zeta\omega\Delta t-\omega^2\Delta t^2} \\ \frac{4\Delta t}{4+4\zeta\omega\Delta t+\omega^2\Delta t^2} & \frac{4+4\zeta\omega\Delta t-\omega^2\Delta t^2}{4+4\zeta\omega\Delta t+\omega^2\Delta t^2} \end{bmatrix}.\end{aligned} \tag{11}$$

Now, Eq. (11) may be used to evaluate numerical stability of the proposed method. As the time or step increases progressively, the condition in which a numerical solution of Eq. (10) be bounded is $\rho(A_m) \leqslant 1$ [29], where $\rho(A_m)$ is the spectral radius of $[A_m]$ defined by $\rho(A_m) = \max|\lambda_i|$, $i = 1,2$ and λ_i the system eigenvalue. Fig. 2 shows the spectral radii for different ζ values as a function of dimensionless time step size, $\Delta t/T$, with system period T. It should be noted that regardless of ζ, the spectral radii are equal to 1 or less than 1 at any time step size. Therefore, the proposed method may be pronounced to be unconditionally stable. The conventional average acceleration Newmark method also has the same unconditional stability [3].

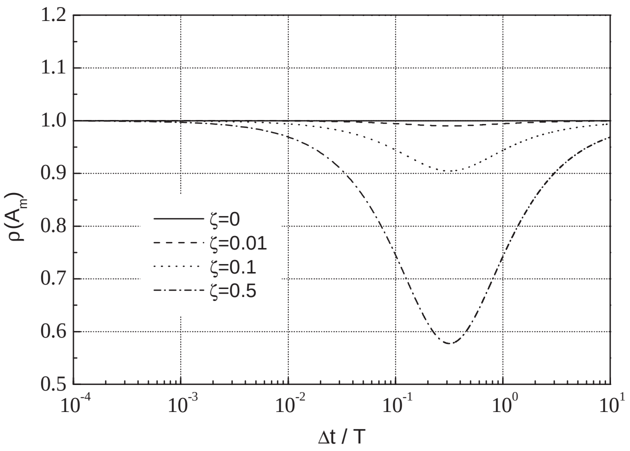

Fig. 2. Spectral radii of the amplification matrix as a function of $\Delta t/T$ for different ζ values.

2.2.2. Accuracy analysis of solution

From Eq. (10) the difference equation of motion between time t_n and t_{n+1} is

$$\{q\}_{n+1} = [A_m]\{q\}_n. \tag{12}$$

The solution of the difference equation is expressed by [6]

$$\{q\}_n = \{r\}\mathrm{e}^{\lambda n \Delta t} = \{r\}\delta^n. \tag{13}$$

Substitution of Eq. (13) into Eq. (12), results in the following standard eigenvalue problem

$$[A_m]\{r\} = \delta\{r\}. \tag{14}$$

The eigenvalues and eigenvectors of Eq. (14) are given by

$$\delta_{1,2} = \frac{4 - \omega^2\Delta t^2 \pm \mathrm{j}4\omega\Delta t\sqrt{1-\zeta^2}}{4 + 4\zeta\omega\Delta t + \omega^2\Delta t^2}, \quad \{r\}_{1,2} = \left\{ \begin{matrix} \omega - \left(\zeta \pm \mathrm{j}\sqrt{1-\zeta^2}\right) \\ 1 \end{matrix} \right\}. \tag{15}$$

The general solution of $\{q_n\}$ can then be written as

$$\{q\}_n = a_1\{r\}_1\delta_1^n + a_2\{r\}_2\delta_2^n. \tag{16}$$

Considering the initial conditions from Eq. (16), the displacement component, x_n, is obtained by

$$x_n = R^n\left(\frac{\dot{x}_0 + x_0\zeta\omega}{\omega\sqrt{1-\zeta^2}}\sin n\phi + x_0 \cos n\phi\right), \tag{17}$$

where $\dot{x}_0$ and x_0 are the initial velocity and displacement, and R and ϕ are given by

$$R = \sqrt{\frac{4 + \omega^2\Delta t^2 - 4\zeta\omega\Delta t}{4 + \omega^2\Delta t^2 + 4\zeta\omega\Delta t}}, \quad \phi = \tan^{-1}\frac{4\omega\Delta t\sqrt{1-\zeta^2}}{4 - \omega^2\Delta t^2}. \tag{18}$$

In addition, an exact solution for a 1-dof under-damped system [30] is given by

$$x(t) = \mathrm{e}^{-\zeta\omega t}\left(\frac{\dot{x}_0 + x_0\zeta\omega}{\omega\sqrt{1-\zeta^2}}\sin\,\omega_d t + x_0\,\cos\,\omega_d t\right), \tag{19}$$

where $\omega_d = \omega\sqrt{1-\zeta^2}$.

The numerical solution of Eq. (17) looks quite similar to the exact solution of Eq. (19). However, an approximate solution by direct-time integration may generally produce an amplitude or period error as illustrated in Fig. 3. Hence, the accuracy of the numerical solution needs to be evaluated more precisely by comparing the vibration periods and amplitudes of the two solutions with each other.

In order to analyze the vibration period, $\sin n\phi$ in Eq. (17) can be rewritten as

$$\sin\frac{\phi}{\Delta t}n\Delta t = \sin\,\omega^* t, \tag{20}$$

where

$$\omega^* = \frac{\phi}{\Delta t} = \frac{1}{\Delta t}\tan^{-1}\frac{4\omega\Delta t\sqrt{1-\zeta^2}}{4-\omega^2\Delta t^2}. \tag{21}$$

Expanding the frequency ω^* by $\omega\Delta t$ as the Taylor series,

$$\omega^* = \omega\sqrt{1-\zeta^2} + \frac{\omega^3\Delta t^2}{12}(4\zeta^2-1)\sqrt{1-\zeta^2} + \mathrm{O}(\omega\Delta t)^5. \tag{22}$$

The first term of Eq. (22) is identical to ω_d, which is the frequency of the exact solution, and the period error is produced by the remaining terms. The period error (%) is defined by

$$e_p = \frac{\bar{T}-T}{T}\times 100\ \ (\%), \tag{23}$$

where $T = 2\pi/\omega_d$ and $\bar{T} = 2\pi/\omega^*$ are the periods of the exact and numerical solutions. Fig. 4 shows the period error as a function of $\Delta t/T$ for different ζ values, and an appropriate Δt can be determined with regard to the period error. The period error has a second-order accuracy for light dampings ($\zeta = 0.01, 0.05, 0.1$) and a fourth-order for large damping ($\zeta = 0.5$) with respect to $\Delta t/T$. It has been reported that the period error for the conventional Newmark method has a second-order accuracy for zero damping [4].

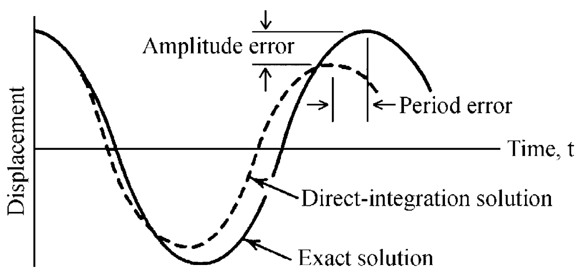

Fig. 3. Possible amplitude and period errors in the numerical solution.

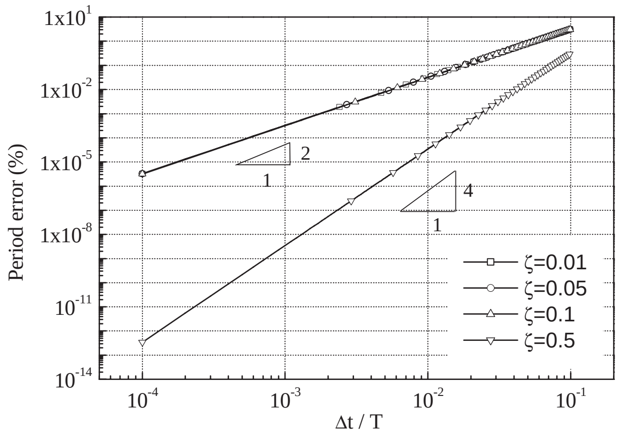

Fig. 4. Period error (%) as a function $\Delta t/T$ of for ζ different values.

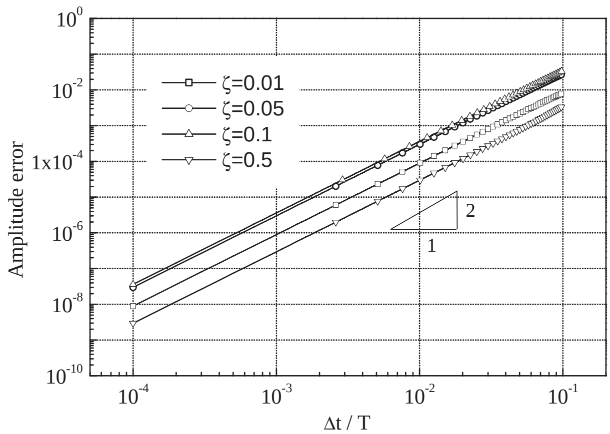

Fig. 5. Amplitude error as a function of $\Delta t/T$ for ζ different values.

On the other hand, expanding the amplitude component, R^n, of Eq. (17) by $\omega\Delta t$ as the Taylor series,

$$R_n = 1 - \zeta(\omega n\Delta t) + \frac{1}{2}\zeta^2(\omega n\Delta t)^2 + \frac{1}{12}\zeta n(3 - 2(2 + n^2)\zeta^2)(\omega\Delta t)^3 + \mathrm{O}(\omega\Delta t)^4. \tag{24}$$

Also, expanding the amplitude component, $\mathrm{e}^{-\zeta\omega t}$, of Eq. (19) as the Taylor series,

$$\mathrm{e}^{-\zeta\omega t} = 1 - \zeta(\omega t) + \frac{1}{2}\zeta^2(\omega t)^2 - \frac{1}{6}\zeta^3(\omega t)^3 + \mathrm{O}(\omega t)^4. \tag{25}$$

From Eqs. (24) and (25) it can be seen that the amplitudes of the numerical and exact solutions are identical up to the third terms and that the remaining terms produce the amplitude error. The amplitude error is defined by

$$e_A = \left|R^n - e^{-\zeta\omega T}\right|. \tag{26}$$

Fig. 5 shows the amplitude error as a function of $\Delta t/T$ for different ζ values, and similarly, an appropriate Δt can be determined concerning the amplitude error. It can be seen that the amplitude error has a second-order accuracy, regardless of damping. In addition, the amplitude error of the conventional Newmark method has a second-order accuracy [4].

3. Transient response analyses

In order to illustrate the utility and validity of the proposed state-space Newmark method, transient response analyses were performed for two sample systems.

3.1. Two-degrees-of-freedom vibration system

A model of a 2-dof vibration system is shown in Fig. 6. For the model, $m_1 = 2000\,\mathrm{kg}$, $m_2 = 50\,\mathrm{kg}$, $k_1 = 10^3\,\mathrm{N/M}$, $k_2 = 10^4\,\mathrm{N/M}$ and the damping ratios are $\zeta_1 = 0.01$ and $\zeta_2 = 0.2$, the

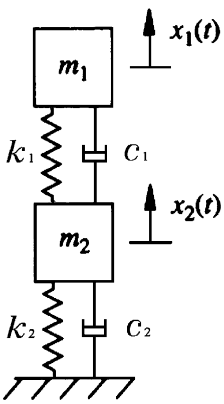

Fig. 6. A model of a 2-dof system.

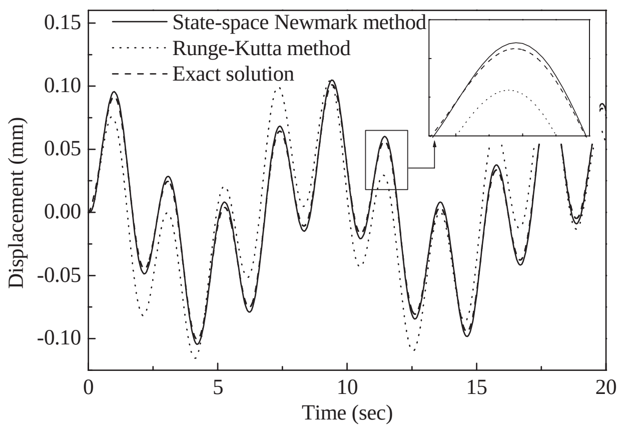

Fig. 7. Transient responses of m_1 obtained by three different methods.

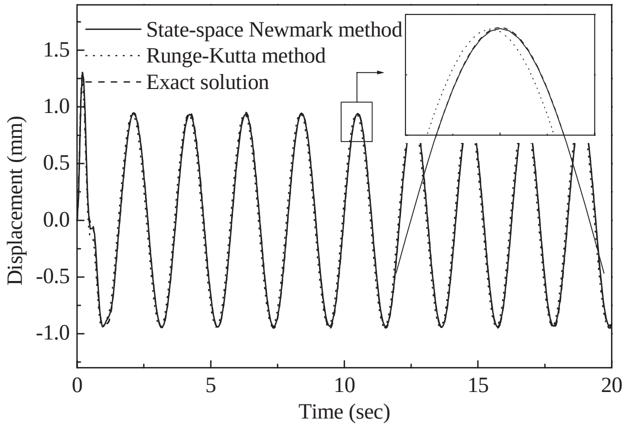

Fig. 8. Transient responses of m_2 obtained by three different methods.

Table 1
Comparison of calculation times using Matlab 6.5

Method	Elapse time (s)
Conventional Newmark	0.032
Proposed state-space Newmark	0.031

external forcing vector $\{f(t)\} = [0,\ 10\cos 3t]^{\mathrm{T}} N$ and the initial conditions $x_1 = x_2 = \dot{x}_1 = \dot{x}_2 = 0$. Figs. 7 and 8 show the transient responses, x_1 and x_2, of m_1 and m_2 obtained by the proposed state-space method, the Runge–Kutta fourth-order method and the exact solution. The time step sizes for numerical analyses have all been set to $\Delta t = 0.05\,\mathrm{s}$. It can be seen that m_1 has a highly vibratory transient response and that m_2 quickly reaches a steady-state oscillating response. For the given Δt, the proposed method yields numerical solutions which are quite close to the exact solutions, regardless of the response characteristic of each mass, and the proposed method also outperforms the Runge–Kutta method.

In addition, for this sample problem, the proposed method and the conventional average acceleration Newmark method provide identical numerical solutions. Although the proposed method has increased row and column sizes twice that of every matrix and column vector involved in the formulation, the two methods require about the same calculation times, as shown in Table 1, since the proposed method involves a more simplified coding and thereby fewer calculation operations.

3.2. *Turbo-compressor rotor-bearing system*

As a practical application sample, a model of a one-stage turbo-compressor rotor-bearing system [31], running at 30,000 rev/min, is shown in Fig. 9, and its detailed finite element modeling

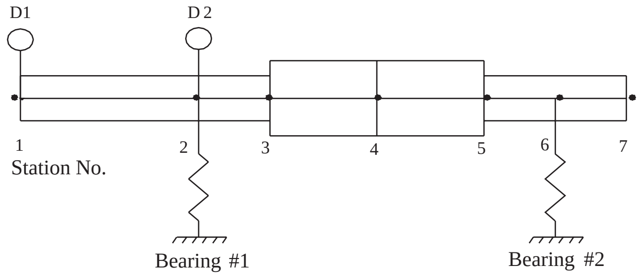

Fig. 9. A finite element model of a one-stage turbo-compressor rotor-bearing system.

Table 2
Finite element modeling data of the turbo-compressor rotor-bearing system

Shaft element		
Element no.	Length (mm)	Diameter (mm)
1	127	38.1
2	50.8	38.1
3	76.2	63.5
4	76.2	63.5
5	50.8	38.1
6	50.8	38.1
$E = 2.0 \times 10^{11}\,\mathrm{N/m^2}$		$\rho = 7833.48\,\mathrm{kg/m^3}$

Disk element			
Disk no.	Mass (kg)	I_t (kg m^2)	I_p (kg m^2)
D1	4.536	0.0183	0.0366
D2	4.536E-05	0.0000	0.0000

Bearing stiffness and damping				
Bearing no.	K_{xx} (N/m)	K_{yy} (N/m)	C_{xx} (N s/m)	C_{yy} (N s/m)
1	2.189E8	2.277E8	8.494E4	8.756E4
2	2.189E8	2.189E8	8.581E4	8.581E4

data are given in Table 2. Unbalances considered for disks 1 and 2 are all 7.2 g mm, and their locations are 0° and 180°, respectively, from a reference point. A transient response analysis of the rotor-bearing system was carried out for the given sudden unbalances, where the time step size was $\Delta t = 0.0001$ s. The horizontal (x-direction) displacement response at station No. 1 obtained by the proposed method for a duration of 0.02 s is shown in Fig. 10. It can be seen that the result obtained by the proposed method is in good agreement with the reference result obtained for the same time step size.

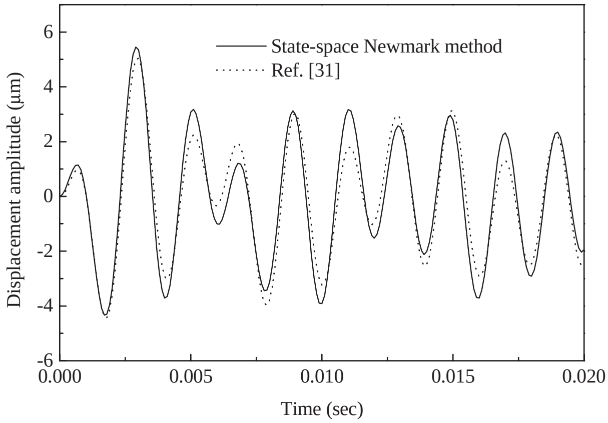

Fig. 10. Horizontal transient response at station No. 1 of the turbo-compressor rotor-bearing system.

4. Conclusions

A state-space Newmark method in which the average velocity concept is introduced to obtain transient responses of general dynamic systems is proposed, and its numerical stability and accuracy are analyzed. The proposed method is a much simpler and more straightforward formulation for obtaining numerical solutions than the conventional average acceleration Newmark method, and as a result, it is more readily coded into a software program. The proposed method is numerically unconditionally stable like the conventional Newmark method, and has a period error of second-order accuracy for small damping and fourth-order for large damping and an amplitude error of second-order, regardless of damping. The utility and validity of the method were confirmed by two application examples. In spite of twice the increase in the row and column sizes of every matrix and column vector involved in the state-space formulation, the proposed method requires almost the same computation time as the conventional Newmark method for the simple problem considered here because of a more simplified coding compared to the latter. Finally, the proposed state-space Newmark method based on the average velocity can be generally applied to the analysis of transient responses of general dynamic systems with a high degree of reliability with respect to its stability and accuracy.

Acknowledgements

This study was supported by the Korea Institute of Machinery and Materials' Basic Project: "Cyber Engineering—A Dynamic Simulation Technology Development" and Doosan Heavy Industry Inc. Endowed Project: "Development of Transient Vibration Response Analysis Technology for Turbine Rotor-Bearing System and Verification Test". These supports are gratefully acknowledged.

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